

Sum of two maximal monotone operators in a general Banach space is maximal

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Abstract

In a real Banach space, we first prove that the sum of a monotone operator of type (FPV) and maximal monotone operator with $\text{dom}A \cap \text{int dom}B \neq \emptyset$ (Rockafellar's constraint qualification) is maximal. This prove leads to the solution of most interesting long-time outstanding problem in monotone operator theory is the sum problem .

Keywords: Sum problem, Fitzpatrick function, maximal monotone operator, monotone operator of type (FPV), Rockaellar's constraint qualification.

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1 Introduction

In monotone operator theory, the most studied and celebrated open problem concerns the maximal monotonicity of the sum of two maximal monotone operators. In 1970, Rockafellar proved it in reflexive space, i.e., the sum of two maximal monotone operators A and B with $\text{dom}A \cap \text{int dom}B \neq \emptyset$ (Rockafellar's constraint qualification) is maximal monotone [11]. Therefore, it remains to study the sum problem in nonreflexive spaces.

In [3], Borwein proves that the sum of two maximal monotone operators A and B is maximal monotone with $\text{int dom}A \cap \text{int dom}B \neq \emptyset$. In [2], Bauschke, Wang and Yao prove that the sum of maximal monotone linear relation and the subdifferential operator of a sublinear function with Rockafellar's constraint qualification is maximal monotone. In [18], Yao extend the results in [2] to the subdifferential operator of any proper lower semicontinuous convex function. Yao [19] proves that the sum of two maximal monotone operators A and B satisfying the conditions $A + N_{\overline{\text{dom}B}}$ is of type (FPV).

In [4], Borwein and Yao prove the maximal monotonicity of the sum of a maximal monotone linear relation and a maximal monotone with the assumptions that $\text{dom}A \cap \text{int dom}B \neq \emptyset$. By relaxing the linearity from the result of [4], Borwein and Yao [6] prove the maximal monotonicity of $A + B$ provided that A and B are maximal monotone operators, $\text{star}(\text{dom}A) \cap \text{int dom}B \neq \emptyset$ and A is of type (FPV). Also in [6] raises a question for further research on relaxing 'starshaped' hypothesis on $\text{dom } A$.

Here, we have given a positive answer to the question posed by Borwein and Yao [6] by relaxing the 'starshaped' domain from [6, Theorem 3.3] which allow us to prove the sum theorem together with the result of [7, Theorem 38]. The remainder of this paper is organized as follows. In Section 2, we provide some auxiliary results and notions which will be used in main results. In section 3,

main results and its corollaries are presented.

2 Preliminaries

Suppose that X is a real Banach space with norm, $\|\cdot\|$ and $\mathbb{U}_X := \{x \in X \mid \|x\| < 1\}$ be the open unit ball in X . X^* is the continuous dual of X , X and X^* are paired by $\langle x, x^* \rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$. A sequence $x_n^* \in X^*$ is said to be *weak** convergence if there is some $x^* \in X^*$ such that $x_n^*(x) \rightarrow x^*(x)$ for all $x \in X$ and we denote it by \rightharpoonup_{w^*} . For a given subset C of X we denote interior of C as $\text{int}C$, closure of C as \overline{C} , boundary of C as $\text{bdry } C$, $\text{conv}C$, $\text{aff}C$ is the convex and affine hull of C . The *intrinsic core* or *relative algebraic interior* of C is denoted by iC [20] and is defined as ${}^iC := \{a \in C \mid \forall x \in \text{aff}(C - C), \exists \delta > 0, \forall \lambda \in [0, \delta] : a + \lambda x \in C\}$. And

$${}^{ic}C := \begin{cases} {}^iC, & \text{if } \text{aff } C \text{ is closed,} \\ \phi, & \text{otherwise} \end{cases}.$$

Also we denote the distance function by $\text{dist}(x, C) := \inf_{c \in C} \|x - c\|$. For any $C, D \subseteq X$, $C - D = \{x - y \mid x \in C, y \in D\}$. For any $\alpha > 0$, $\alpha C := \{\alpha x \mid x \in C\}$. Let $A : X \rightrightarrows X^*$ be a set-valued operator (also known as multifunction or point-to-set mapping) from X to X^* , i.e., for every $x \in X$, $Ax \subseteq X^*$. Domain of A is denoted as $\text{dom}A := \{x \in X \mid Ax \neq \phi\}$ and range of A is $\text{ran}A = \{x^* \in X^* \mid x^* \in Ax \text{ for some } x \in \text{dom}A\}$. Graph of A is denoted as $\text{gra}A = \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$. A is said to be linear relation if $\text{gra}A$ is a linear subspace. The set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \text{gra}A.$$

Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$ we say that (x, x^*) is monotonically related to $\text{gra}A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \text{gra}A.$$

And a set valued mapping A is said to maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). In other words A is maximal monotone if for any $(x, x^*) \in X \times X^*$ is monotonically related to $\text{gra}A$ then $(x, x^*) \in \text{gra}A$. We say that A is of type (FPV) if for every open set $U \subseteq X$ such that $U \cap \text{dom}A \neq \emptyset$, $x \in U$ and (x, x^*) is monotonically related to $\text{gra}A \cap U \times X^*$, then $(x, x^*) \in \text{gra}A$. Every monotone operators of type (FPV) are maximal monotone operators [14].

Let $f : X \rightarrow]-\infty, +\infty]$ be a function and its domain is defined as $\text{dom}f := f^{-1}(\mathbb{R})$. f is said to be proper if $\text{dom}f \neq \emptyset$. Let f be any proper convex function then the subdifferential operator of f is defined as $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* \mid \langle y - x, x^* \rangle + f(x) \leq f(y), \forall y \in X\}$. Subdifferential operators are of type (FPV)[14]. For every $x \in X$, the normal cone operator at x is defined by $N_C(x) = \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \emptyset$, if $x \notin C$. Also it may be verified that the normal cone operator is of type (FPV) [14]. For $x, y \in X$, we denote $[x, y] := \{tx + (1 - t)y \mid 0 \leq t \leq 1\}$ and star or center of C as $\text{star}C := \{x \in C \mid [x, c] \subseteq C, \quad \forall c \in C\}$ [20].

We denote the projection map by $P_X : X \times X^* \rightarrow X$ by $P_X(x, x^*) = x$. For any two A and B monotone operators, the sum operator is defined as $A + B : X \rightrightarrows X^* : x \mapsto Ax + Bx = \{a^* + b^* \mid a^* \in Ax \text{ and } b^* \in Bx\}$. It may be checked that $A + B$ is monotone.

Fact 2.1. [9, Theorem 2.28] *Let $A : X \rightrightarrows X^*$ be monotone with $\text{int dom}A \neq \emptyset$. Then A is locally bounded at $x \in \text{int dom}A$, i.e., there exist $\delta > 0$ and $K > 0$*

such that

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta \mathbb{U}_X) \cap \text{dom} A.$$

Fact 2.2 (Fitzpatrick). [8, Corollary 3.9] Let $A : X \rightrightarrows X^*$ be maximal monotone, and $F_A : X \times X^* \rightarrow (-\infty, +\infty]$ defined by

$$F_A(x, x^*) = \sup_{(a, a^*) \in \text{gra} A} (\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle),$$

which is the Fitzpatrick function associated with A . Then for every $(x, x^*) \in X \times X^*$, the inequality $\langle x, x^* \rangle \leq F_A(x, x^*)$ is true, and equality holds if and only if $(x, x^*) \in \text{gra} A$.

Fact 2.3. [15, Theorem 3.4 and Corollary 5.6], or [14, Theorem 24.1(b)] Let $A, B : X \rightrightarrows X^*$ be maximal monotone operator. Assume $\bigcup_{\lambda > 0} \lambda[P_X(\text{dom} F_A) - P_X(\text{dom} F_B)]$ is a closed subspace. If $F_{A+B} \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$, then $A + B$ is maximal monotone.

Fact 2.4. [20, Theorem 1.1.2(ii)] Let C be a convex subset of X . If $a \in \text{int} C$ and $x \in \overline{C}$, then $[a, x[\subset \text{int} C$.

Fact 2.5 (Rockafellar). [10, Theorem 1] or [14, Theorem 27.1 and Theorem 27.3] Let $A : X \rightrightarrows X^*$ be maximal monotone with $\text{int dom} A \neq \emptyset$. Then $\text{int dom} A = \overline{\text{int dom} A}$; and $\text{int dom} A$ and $\overline{\text{dom} A}$ is convex.

Fact 2.6 (Norm-weak* closed graph). [5, Corollary 4.1] Let $A : X \rightrightarrows X^*$ be maximally monotone such that $\text{int dom} A \neq \emptyset$. Then $\text{gra} A$ is norm \times weak* closed.

Fact 2.7. [14, Theorem 48.4(d)] Let $f : X \rightarrow]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then ∂f is of type (FPV).

Fact 2.8. [6, Proposition 3.1] Let $A : X \rightrightarrows X^*$ be of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that $\text{dom} A \cap \text{int dom} B \neq \emptyset$. Let $(z, z^*) \in X \times X^*$ with $z \in \overline{\text{dom} B}$. Then $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$.

Fact 2.9 (Simons). [14, Theorem 46.1] Let X be a nonzero Banach space, $A : X \rightrightarrows X^*$ be maximally monotone with convex graph. Then A is of type (FPV).

Fact 2.10 (Simons and Verona-Verona). [14, Theorem 44.1] Let $A : X \rightrightarrows X^*$ be maximally monotone. Suppose that for every closed convex subset C of X with $\text{dom}A \cap \text{int}C \neq \emptyset$, the operator $A + N_C$ is maximally monotone. Then A is of type (FPV).

Fact 2.11. [1, Lemma 2.5] Let C be a nonempty closed convex subset of X such that $\text{int}C \neq \emptyset$. Let $c_0 \in \text{int}C$ and suppose that $z \in X \setminus C$. Then there exists $\lambda \in]0, 1[$ such that $\lambda c_0 + (1 - \lambda)z \in \text{bdry } C$.

Fact 2.12. [14, Theorem 44.2] Let $A : X \rightrightarrows X^*$ be of type (FPV). Then

$$\overline{\text{dom}A} = \overline{\text{conv}(\text{dom}A)} = \overline{P_X(\text{dom}F_A)}.$$

Fact 2.13. [6, Lemma 2.10] Let $A : X \rightrightarrows X^*$ be monotone, and Let $B : X \rightrightarrows X^*$ be maximally monotone. Let $(z, z^*) \in X \times X^*$. Suppose $x_0 \in \text{dom}A \cap \text{int dom}B$ and that there exists a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra}A \cap (\text{dom}B \times X^*)$ such that $(a_n)_{n \in \mathbb{N}}$ converges to a point in $[x_0, z[$, while $\langle z - a_n, a_n^* \rangle \rightarrow \infty$. Then $F_{A+B}(z, z^*) = +\infty$.

Fact 2.14. [6, Lemma 2.12] Let $A : X \rightrightarrows X^*$ be of type (FPV). Suppose $x_0 \in \text{dom}A$ but that $z \notin \overline{\text{dom}A}$. Then there exists a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra}A$ so that $(a_n)_{n \in \mathbb{N}}$ converges to a point in $[x_0, z[$ and $\langle z - a_n, a_n^* \rangle \rightarrow +\infty$.

Fact 2.15. [The Banach-Alaoglu Theorem]/[12, Theorem 3.15] The closed unit ball in X^* , B_X^* is weak star compact.

Fact 2.16. [19] Let $A : X \rightrightarrows X^*$ be maximally monotone and $z \in \overline{\text{dom}A} \setminus \text{dom}A$. Then for every sequence $(z_n)_{n \in \mathbb{N}}$ in $\text{dom}A$ such that $z_n \rightarrow z$, we have $\lim_{n \rightarrow \infty} \inf \|A(z_n)\| = +\infty$.

Proof. Suppose to the contrary that there exists a sequence $z_{n_k}^* \in A(z_{n_k})$ and $L > 0$ such that $\sup_{k \in \mathbb{N}} \|z_{n_k}^*\| \leq L$. By Fact 2.15, there exists a weak* convergent subnet, $(z_\beta^*)_{\beta \in J}$ of $z_{n_k}^*$ such that $z_\beta^* \rightharpoonup_w^* z_\infty^* \in X^*$. By [5, Fact 3.5], we have $(z, z_\infty^*) \in \text{gra}A$, which is a contradiction to our assumption that $z \notin \text{dom}A$. \square

Fact 2.17. [6] Let $A : X \rightrightarrows X^*$ be of type (FPV), and Let $B : X \rightrightarrows X^*$ be maximally monotone. Let $(z, z^*) \in X \times X^*$. Suppose $x_0 \in \text{dom}A \cap \text{int dom}B$. Assume that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $\text{dom}A \cap \text{dom}B$ and $\beta \in [0, 1]$ such that $a_n \rightarrow \beta z + (1 - \beta)x_0$ and $a_n \in \text{bdry dom}B$ Then $F_{A+B}(z, z^*) = +\infty$.

Fact 2.18. [19] Let $A : X \rightrightarrows X^*$ be of type (FPV), and Let $B : X \rightrightarrows X^*$ be maximally monotone. Let $(z, z^*) \in X \times X^*$. Suppose $x_0 \in \text{dom}A \cap \text{int dom}B$. Assume that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $\text{dom}A \cap [\overline{\text{dom}B} \setminus \text{dom}B]$ and $\beta \in [0, 1]$ such that $a_n \rightarrow \beta z + (1 - \beta)x_0$ Then $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$.

Fact 2.19. [FPV and convex closure Theorem][7, Theorem 38] If A is maximal monotone on a Banach space, then A is of type (FPV). In particular, $\text{dom}A$ has a convex closure.

3 Main Results

We first prove the useful results which play an important role in proving our main results.

Lemma 3.1. Let $A : X \rightrightarrows X^*$ be of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone. Let $(z, z^*) \in X \times X^*$, $x_0 \in \text{dom}A \cap \text{int dom}B$. Assume that there exists $(a_n)_{n \in \mathbb{N}} \in \overline{\text{dom}A} \cap \text{bdry } \overline{\text{dom}B}$ such that it converges to a point in $[x_0, z]$. Then $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$.

Proof. Assume to the contrary

$$F_{A+B}(z, z^*) < \langle z, z^* \rangle. \quad (1)$$

By the necessary translation if necessary, we can suppose that $x_0 = 0 \in \text{dom}A \cap \text{int dom}B$ and $(0, 0) \in \text{gra}A \cap \text{gra}B$. By the assumption that, there exists $0 \leq \beta \leq 1$ such that

$$a_n \longrightarrow \beta z. \quad (2)$$

Since $0 \in \text{int dom}B$, by (1) and Fact 2.8, we have

$$0 < \beta < 1 \quad \text{and} \quad \beta z \neq 0. \quad (3)$$

We set

$$y_0 := \beta z. \quad (4)$$

By $0 \in \text{int dom}B$, there exists $0 < \rho_0$ such that

$$\rho_0 \mathbb{U}_{\mathbb{X}} \subseteq \text{dom}B. \quad (5)$$

Now we show that there exists $\beta \leq \delta_n \in [1 - \frac{1}{n}, 1[$ such that

$$H_n \subseteq \text{dom}B \quad (6)$$

where

$$H_n := \delta_n \beta z + (1 - \delta_n) \rho_0 \mathbb{U}_{\mathbb{X}}. \quad (7)$$

By Fact 2.4 and Fact 2.5, we have for every $s \in (0, 1)$,

$$s\beta z + (1 - s)\rho_0 \mathbb{U}_{\mathbb{X}} \subseteq \overline{\text{int dom}B} = \text{int dom}B.$$

Hence (6) holds. Since $a_n \rightarrow y_0$ and $\delta_n \beta z = v_n$ (say) by (7), $v_n \rightarrow y_0$. Then we can suppose that

$$\|v_n\| \leq \|y_0\| + 1 \leq \|z\| + 1, \quad \forall n \in \mathbb{N} \text{ (by (4))}. \quad (8)$$

Next we show that there exists $(\tilde{a}_n, \tilde{a}_n^*)_{n \in \mathbb{N}}$ in $\text{gra}A \cap (H_n \times X^*)$ such that

$$\langle z - \tilde{a}_n, \tilde{a}_n^* \rangle \geq -K_0 \|a_n^*\| \quad (9)$$

where $K_0 = \frac{1}{\beta^2}(2\|z\| + 2)$. Since $\delta_n\beta z = v_n \in H_n$ and $a_n^* \in X^*$, then we consider two cases.

Case 1: $(v_n, a_n^*) \in \text{gra}A$.

Take $(\tilde{a}_n, \tilde{a}_n^*) := (v_n, a_n^*)$.

$$\begin{aligned} \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle &= \langle z - v_n, a_n^* \rangle \\ &\geq -\|z - v_n\| \|a_n^*\| \end{aligned} \tag{10}$$

$$\geq -(2\|z\| + 2)\|a_n^*\| \quad \text{by equation(8)}. \tag{11}$$

$$\geq -K_0\|a_n^*\|. \tag{12}$$

Hence (9) holds.

Case 2: $(v_n, a_n^*) \notin \text{gra}A$.

By Fact 2.12 and by the assumption $a_n \in \overline{\text{dom}A}$, we get $v_n = \delta_n\beta z \in \overline{\text{dom}A}$. Therefore, $H_n \cap \text{dom}A \neq \emptyset$. Since $(v_n, a_n^*) \notin \text{gra}A$ and $v_n \in H_n$, by using (FPV) property of A , there exists $(\tilde{a}_n, \tilde{a}_n^*) \in \text{gra}A \cap (H_n \times X^*)$ such that

$$\langle v_n - \tilde{a}_n, a_n^* - \tilde{a}_n^* \rangle < 0.$$

Thus, we have

$$\begin{aligned} &\langle v_n - \tilde{a}_n, \tilde{a}_n^* - a_n^* \rangle > 0 \\ \Rightarrow &\langle v_n - \tilde{a}_n, \tilde{a}_n^* \rangle > \langle v_n - \tilde{a}_n, a_n^* \rangle \\ \Rightarrow &\langle \delta_n\beta z - \delta_n\beta \tilde{a}_n + \delta_n\beta \tilde{a}_n - \tilde{a}_n, \tilde{a}_n^* \rangle > \langle v_n - \tilde{a}_n, a_n^* \rangle \\ \Rightarrow &\langle \delta_n\beta(z - \tilde{a}_n) - (1 - \delta_n\beta)\tilde{a}_n, \tilde{a}_n^* \rangle > \langle v_n - \tilde{a}_n, a_n^* \rangle \\ \Rightarrow &\langle \delta_n\beta(z - \tilde{a}_n), \tilde{a}_n^* \rangle > (1 - \delta_n\beta)\langle \tilde{a}_n, \tilde{a}_n^* \rangle + \langle v_n - \tilde{a}_n, a_n^* \rangle. \end{aligned}$$

Since $\beta \leq \delta_n < 1$, $(0, 0) \in \text{gra}A$ and $(\tilde{a}_n, \tilde{a}_n^*) \in \text{gra}A$, applying monotonicity of

A , we have

$$\begin{aligned}
& \langle \delta_n \beta (z - \tilde{a}_n), \tilde{a}_n^* \rangle \geq \langle v_n - \tilde{a}_n, a_n^* \rangle \\
& \Rightarrow \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle \geq \frac{1}{\delta_n \beta} \langle v_n - \tilde{a}_n, a_n^* \rangle. \\
& \Rightarrow \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle \geq -\frac{1}{\delta_n \beta} \|v_n - \tilde{a}_n\| \|a_n^*\| \\
& \Rightarrow \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle \geq -\frac{1}{\beta^2} \|v_n - \tilde{a}_n\| \|a_n^*\|. \tag{13}
\end{aligned}$$

Since $v_n, \tilde{a}_n \in H_n$, then we have $\tilde{a}_n \rightarrow y_0$ and we can suppose that

$$\|\tilde{a}_n\| \leq \|y_0\| + 1 \leq \|z\| + 1, \quad \forall n \in \mathbb{N}. \tag{14}$$

Appealing to equation (13), we have

$$\begin{aligned}
\langle z - \tilde{a}_n, \tilde{a}_n^* \rangle & \geq -\frac{1}{\beta^2} (2\|z\| + 2) \|a_n^*\| \\
& = -K_0 \|a_n^*\|.
\end{aligned}$$

Hence (9) holds.

Since $\beta z \in \text{bdry dom } B$ then we consider two cases.

Case 1. $\beta z \notin \text{dom } B$.

By Fact 2.16, we have

$$\inf \|B(H_n)\| \geq K_0 \|a_n^*\| n. \tag{15}$$

Since $\tilde{a}_n \in H_n$, equation (6) implies that $\tilde{a}_n \in \text{int dom } B$ and $\tilde{a}_n \in \text{dom } A$. Again since $\tilde{a}_n \in H_n$ then take $b_n^* \in B(\tilde{a}_n)$ by (15),

$$\|b_n^*\| \geq n K_0 \|a_n^*\|. \tag{16}$$

We compute

$$\begin{aligned}
F_{A+B}(z, z^*) & = \sup_{\{\tilde{a}_n^* + b_n^* \in (A+B)(\tilde{a}_n)\}} [\langle \tilde{a}_n, z^* \rangle + \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle + \langle z - \tilde{a}_n, b_n^* \rangle] \\
& \geq [\langle \tilde{a}_n, z^* \rangle + \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle + \langle z - \tilde{a}_n, b_n^* \rangle]. \tag{17}
\end{aligned}$$

By (9), we have

$$\begin{aligned}
F_{A+B}(z, z^*) &\geq [\langle \tilde{a}_n, z^* \rangle + \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle + \langle z - \tilde{a}_n, b_n^* \rangle] \\
\Rightarrow F_{A+B}(z, z^*) &\geq \langle \tilde{a}_n, z^* \rangle - K_0 \|a_n^*\| + \langle z - \tilde{a}_n, b_n^* \rangle \\
\Rightarrow \frac{F_{A+B}(z, z^*)}{\|b_n^*\|} &\geq \left\langle \tilde{a}_n, \frac{z^*}{\|b_n^*\|} \right\rangle - \frac{K_0 \|a_n^*\|}{\|b_n^*\|} + \left\langle z - \tilde{a}_n, \frac{b_n^*}{\|b_n^*\|} \right\rangle. \tag{18}
\end{aligned}$$

By Banach-Alaoglu Theorem [12, Theorem 3.15], there exist a *weak** convergent subnet $(\frac{b_\gamma^*}{\|b_\gamma^*\|})$ of $(\frac{b_n^*}{\|b_n^*\|})$ such that

$$\frac{b_\gamma^*}{\|b_\gamma^*\|} \longrightarrow v_\infty^* \in X^*. \tag{19}$$

Using (19), (16) and taking limit in (18) along the subnet, we have $\langle z - \beta z, v_\infty^* \rangle \leq 0$

$$\langle z, v_\infty^* \rangle \leq 0. \tag{20}$$

On the other hand, since $0 \in \text{int dom } B$ by using Fact 2.1, there exist $\epsilon > 0$ and $M > 0$ such that

$$\sup_{y^* \in By} \|y^*\| \leq M, \quad \forall y \in \epsilon \mathbb{U}_X. \tag{21}$$

Since $(\tilde{a}_n, b_n^*) \in \text{gra } B$, then we have

$$\begin{aligned}
&\langle \tilde{a}_n - y, b_n^* - y^* \rangle \geq 0, \quad \forall y \in \epsilon \mathbb{U}_X, y^* \in B(y), n \in \mathbb{N} \\
&\Rightarrow \langle \tilde{a}_n, b_n^* \rangle - \langle y, b_n^* \rangle + \langle \tilde{a}_n - y, -y^* \rangle \geq 0 \quad \forall y \in \epsilon \mathbb{U}_X, y^* \in B(y), n \in \mathbb{N} \\
&\Rightarrow \langle \tilde{a}_n, b_n^* \rangle - \langle y, b_n^* \rangle \geq -(\|\tilde{a}_n\| + \epsilon)M, \quad \forall y \in \epsilon \mathbb{U}_X, n \in \mathbb{N} \\
&\Rightarrow \langle \tilde{a}_n, b_n^* \rangle \geq \epsilon \|b_n^*\| - (\|\tilde{a}_n\| + \epsilon)M, \quad \forall n \in \mathbb{N} \\
&\Rightarrow \left\langle \tilde{a}_n, \frac{b_n^*}{\|b_n^*\|} \right\rangle \geq \epsilon - \frac{(\|\tilde{a}_n\| + \epsilon)M}{\|b_n^*\|}, \quad \forall n \in \mathbb{N}. \tag{22}
\end{aligned}$$

Using (19) and taking limit in (22) along the subnet, we have $\langle \beta z, v_\infty^* \rangle \geq \epsilon > 0$ which contradict to (20).

Case 2. $\beta z \in \text{dom } B$.

Since $\beta z \in \text{bdry dom } B$. Take $y_0^* \in N_{\overline{\text{dom } B}}(\beta z)$ such that

$$\langle y_0^*, \beta z - y \rangle > 0, \quad \text{for every } y \in \text{int dom } B. \quad (23)$$

Thus, $ty_0^* \in N_{\overline{\text{dom } B}}(\beta z), \forall t > 0$. Since $\beta z \in \overline{\text{dom } A}$, we again consider the following two subcases:

Subcases 2.1 $\beta z \in \text{dom } A$.

Since $0 \in \text{int dom } B$ then by (23), we have

$$\langle y_0^*, z \rangle > 0. \quad (24)$$

Since B is maximally monotone, $B = B + N_{\overline{\text{dom } B}}$ and $\beta z \in \text{dom } A \cap \text{dom } B$. Then we compute

$$F_{A+B}(z, z^*) \geq \sup[\langle z - \beta z, A(\beta z) \rangle + \langle z - \beta z, B(\beta z) + ty_0^* \rangle + \langle z^*, \beta z \rangle].$$

Thus,

$$\frac{F_{A+B}(z, z^*)}{t} \geq \sup \left[\langle z - \beta z, \frac{A(\beta z)}{t} \rangle + \langle z - \beta z, \frac{B(\beta z)}{t} + y_0^* \rangle + \frac{\langle z^*, \beta z \rangle}{t} \right].$$

By (1), letting $t \rightarrow \infty$ we have $\langle z - \beta z, y_0^* \rangle \leq 0$ and since $\beta < 1$ we obtain

$$\langle z, y_0^* \rangle \leq 0,$$

which contradicts (24).

Subcases 2.2 $\beta z \notin \text{dom } A$.

Set

$$U_n = \beta z + \frac{1}{n} \mathbb{U}_{\mathbb{X}}. \quad (25)$$

By the assumption $\beta z \in \overline{\text{dom } A}$, Therefore, $U_n \cap \text{dom } A \neq \emptyset$. Since $(\beta z, ty_0^*) \notin \text{gra } A$ and $\beta z \in U_n$, by using (FPV) property of A , there exists $(\tilde{a}_n, \tilde{a}_n^*) \in \text{gra } A \cap (U_n \times X^*)$ such that

$$\langle \beta z - \tilde{a}_n, ty_0^* - \tilde{a}_n^* \rangle < 0.$$

$$\begin{aligned}
& \langle \beta z - \tilde{a}_n, \tilde{a}_n^* \rangle > t \langle \beta z - \tilde{a}_n, y_0^* \rangle \\
& \Rightarrow \langle \beta(z - \tilde{a}_n) - (1 - \beta)\tilde{a}_n, \tilde{a}_n^* \rangle > t \langle \beta z - \tilde{a}_n, y_0^* \rangle \\
& \Rightarrow \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle > \frac{(1 - \beta)}{\beta} \langle \tilde{a}_n, \tilde{a}_n^* \rangle + \frac{t}{\beta} \langle \beta z - \tilde{a}_n, y_0^* \rangle \\
& \Rightarrow \langle z - \tilde{a}_n, \tilde{a}_n^* \rangle > \frac{t}{\beta} \langle \beta z - \tilde{a}_n, y_0^* \rangle \quad (\text{by monotonicity of } A). \tag{26}
\end{aligned}$$

Since $\beta z \in \text{bdry dom} B$. By $0 \in \text{int dom} B$ and Fact 2.5, we have $\beta z \in \text{int}(1 + \epsilon) \text{dom} B$, for every $0 < \epsilon < 1$. Since $\tilde{a}_n \in U_n$, (25) implies $\tilde{a}_n \rightarrow \beta z$. Thence, there exists $n_0 \in \mathbb{N}$ such that $\tilde{a}_n \in \text{int}(1 + \epsilon) \text{dom} B$, $\forall n \geq n_0$. Thus, for every $0 < \epsilon < 1$, $\tilde{a}_n \in \text{int}(1 + \epsilon) \text{dom} B$, $\forall n \geq n_0$. Therefore, $\tilde{a}_n \in \overline{\text{dom} B} \forall n \geq n_0$. By Fact 2.18, we have $\tilde{a}_n \in \text{dom} B$. Since $a_n \in \text{dom} A$ and $\tilde{a}_n \rightarrow \beta z$ then by Fact 2.17, we have $\tilde{a}_n \in \text{int dom} B$. By using Fact 2.1, there exists, $\rho > 0$ and $K > 0$ such that

$$\sup_{c_n^* \in B(c_n)} \|c_n^*\| \leq K, \quad \forall c_n \in \tilde{a}_n + \rho \mathbb{U}_{\mathbb{X}}. \tag{27}$$

Again since $\tilde{a}_n \in \text{int dom} B$ by 23, we have $\langle y_0^*, \beta z - \tilde{a}_n \rangle > 0$. Therefore, by (26) and (27), we obtain

$$\begin{aligned}
F_{A+B}(z, z^*) & \geq \sup_{b_n^* \in B(\tilde{a}_n)} [\langle z - \tilde{a}_n, \tilde{a}_n^* \rangle + \langle z - \tilde{a}_n, b_n^* \rangle + \langle z^*, \tilde{a}_n \rangle], \quad \forall n \geq n_0 \\
& \geq \sup \left[\frac{t}{\beta} \langle \beta z - \tilde{a}_n, y_0^* \rangle - \|z - \tilde{a}_n\| \|b_n^*\| - \|z^*\| \|\tilde{a}_n\| \right] \\
& \geq \sup \left[\frac{t}{\beta} \langle \beta z - \tilde{a}_n, y_0^* \rangle - (2\|z\| + 1)K - (\|z\| + 1)\|z^*\| \right] \quad (\text{by (14)}).
\end{aligned}$$

Letting $t \rightarrow +\infty$, we have $F_{A+B}(z, z^*) = +\infty$.

Hence by combining all the above cases we have $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$. \square

Proposition 3.2. *Let $A : X \rightrightarrows X^*$ be of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone with $\text{dom} A \cap \text{int dom} B \neq \emptyset$. Suppose that there exists $(z, z^*) \in X \times X^*$ such that $F_{A+B}(z, z^*) < \langle z, z^* \rangle$. Then $z \in \overline{\text{dom} A}$.*

Proof. By the necessary translation if necessary, we can suppose that $0 \in \text{dom} A \cap \text{int dom} B$ and $(0, 0) \in \text{gra} A \cap \text{gra} B$. We assume to the contrary that

$$z \notin \overline{\text{dom} A}. \tag{28}$$

By using equation (28) and Fact 2.14, we have there exist $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra}A$ and $0 \leq \lambda < 1$ such that

$$\langle z - a_n, a_n^* \rangle \longrightarrow +\infty \quad \text{and} \quad a_n \longrightarrow \lambda z. \quad (29)$$

Now we consider the following cases.

Case 1: There exists a subsequence of $(a_n)_{n \in \mathbb{N}}$ in $\text{dom}B$.

We can suppose that $a_n \in \text{dom}B$ for every $n \in \mathbb{N}$. Thus by 29 and Fact 2.13, we have $F_{A+B}(z, z^*) = +\infty$, which is a contradiction to the hypothesis that $F_{A+B}(z, z^*) < \langle z, z^* \rangle$.

Case 2: There exists $n_1 \in \mathbb{N}$ such that $a_n \notin \text{dom}B$ for every $n \geq n_1$.

Now we suppose that $a_n \notin \text{dom}B$ for every $n \in \mathbb{N}$. Since $a_n \notin \text{dom}B$, by Fact 2.5 and Fact 2.11, there exists $\beta_n \in [0, 1]$ such that

$$\beta_n a_n \in \text{bdry } \overline{\text{dom}B}. \quad (30)$$

By equation (29), we can suppose that

$$\beta_n a_n \longrightarrow \beta z \quad (31)$$

Since $0 \in \text{int } \text{dom}B$ then by (28) and Fact 2.12, we have

$$0 < \beta < 1. \quad (32)$$

By (30), Fact 2.12 and Lemma 3.1, we have $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$, which is a contradiction. Hence by combining all the above cases, we have proved that $z \in \overline{\text{dom}A}$. \square

Theorem 3.3 (Main Result). *Let $A, B : X \rightrightarrows X^*$ be maximally monotone with $\text{dom}A \cap \text{int } \text{dom}B \neq \emptyset$. Assume that A is of type (FPV). Then $A+B$ is maximally monotone.*

Proof. By the necessary translation if necessary, we can suppose that $0 \in \text{dom}A \cap \text{int dom}B$ and $(0, 0) \in \text{gra}A \cap \text{gra}B$. From Fact 2.2, we have $\text{dom}A \subseteq P_X(\text{dom}F_A)$ and $\text{dom}B \subseteq P_X(\text{dom}F_B)$. Hence

$$\bigcup_{\lambda > 0} \lambda(P_X(\text{dom}F_A) - P_X(\text{dom}F_B)) = X.$$

Thus, by Fact 2.3 it is sufficient to prove that

$$F_{A+B}(z, z^*) \geq \langle z, z^* \rangle, \quad \forall (z, z^*) \in X \times X^*. \quad (33)$$

Let $(z, z^*) \in X \times X^*$. On the contrary assume that

$$F_{A+B}(z, z^*) < \langle z, z^* \rangle. \quad (34)$$

Then by equation (34) Proposition 3.2 and Fact 2.8 we have

$$z \in \overline{\text{dom}A} \setminus \overline{\text{dom}B}. \quad (35)$$

Since $z \in \overline{\text{dom}A}$, there exists $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra}A$ such that

$$a_n \longrightarrow z. \quad (36)$$

By (35), $a_n \notin \overline{\text{dom}B}$ for all but finitely many terms a_n . We can suppose that $a_n \notin \overline{\text{dom}B}$ for all $n \in \mathbb{N}$. By Fact 2.5 and Fact 2.11, there exists $\beta_n \in]0, 1[$ such that

$$\beta_n a_n \in \text{bdry } \overline{\text{dom}B}. \quad (37)$$

By (36) and $\beta \in [0, 1]$ we have

$$\beta_n a_n \longrightarrow \beta z. \quad (38)$$

By (37) and (35) we have $0 < \beta < 1$. By (37), Fact 2.12 and Lemma 3.1, we have a contradiction. Thus, we have $F_{A+B}(z, z^*) \geq \langle z, z^* \rangle$ for all $(z, z^*) \in X \times X^*$. Hence $A + B$ is maximally monotone. \square

The following corollary provides an affirmative answer to the question first posed by S. Simons in his monograph [13, Problem 41.4] i.e.,

Let $A : X \rightrightarrows X^$ be a monotone operator of type (FPV) and let C be a closed convex subset of X such that $\text{dom}A \cap \text{int } C \neq \emptyset$. Is $A + N_C$ maximally monotone?*

Corollary 3.4. *Let $A : X \rightrightarrows X^*$ be a monotone operator of type (FPV) and let C be a closed convex subset of X such that $\text{dom}A \cap \text{int } C \neq \emptyset$. Then $A + N_C$ is maximally monotone.*

Proof. Take $B = N_C$ and directly apply Theorem 3.3. \square

Finally, we have achieved our goal in the following which was the longtime outstanding problem in Monotone operator theory.

Corollary 3.5 (Sum Theorem). *Let $A, B : X \rightrightarrows X^*$ be maximally monotone with $\text{dom}A \cap \text{int } \text{dom}B \neq \emptyset$. Then $A + B$ is maximally monotone.*

Proof. By Fact 2.19, A is of type (FPV). Then directly apply Theorem 3.3, we have $A + B$ is maximal monotone operator. \square

Corollary 3.6 (Linear relation). *[4, Theorem 3.1] Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that $\text{dom}A \cap \text{int } \text{dom}B \neq \emptyset$. Then $A + B$ is maximally monotone.*

Proof. By Fact 2.9, we obtain A is of type (FPV) and then directly apply Theorem 3.3. \square

In the following we obtain the result of [6, Corollary 4.4] as a corollary to our main result.

Corollary 3.7 (Nonempty interior). *[3, Theorem 9(i)] Let $A, B : X \rightrightarrows X^*$ be maximally monotone with $\text{int } \text{dom}A \cap \text{int } \text{dom}B \neq \emptyset$. Then $A + B$ is maximally monotone.*

Proof. Apply directly Corollary 3.5. □

The following result has been proved in [6, Corollary 4.6] with the assumption that $\text{dom}A$ is convex and here we obtain this result by dropping the convexity hypothesis.

Corollary 3.8 (FPV property of the sum). *Let $A, B : X \rightrightarrows X^*$ be maximally monotone with $\text{dom}A \cap \text{int dom}B \neq \emptyset$. Then $A + B$ is type (FPV)*

Proof. By Corollary 3.5, $A + B$ is maximal monotone. Hence Fact 2.19 implies $A + B$ is of type (FPV). □

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